ANOMALOUS HALL EFFECT AND SPIN HALL EFFECT (theory)

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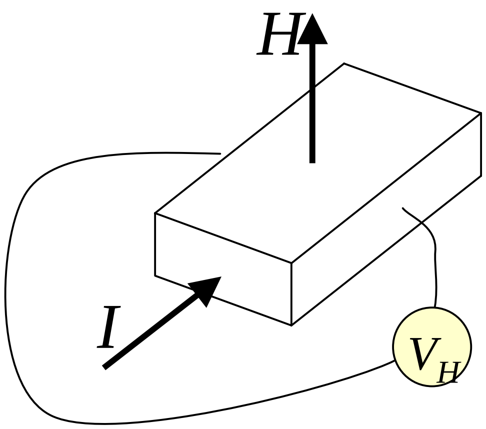
Outline:

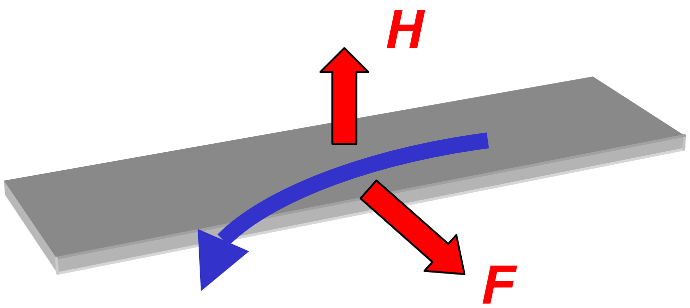
- Introduction
- Spin-orbit interaction and electron scattering
- Electron transport within Kubo formalism
- Mechanisms of anomalous Hall effect
- Anomalous Hall effect and topology
- Effect of impurities
- Spin currents
- Generation of spin currents: spin Hall effect
- Quantized anomalous and spin Hall effects
- Conclusions

Introduction

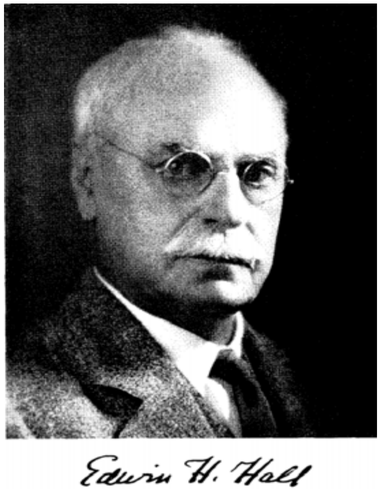
Ordinary Hall effect



$$\rho_H = R_0 H$$

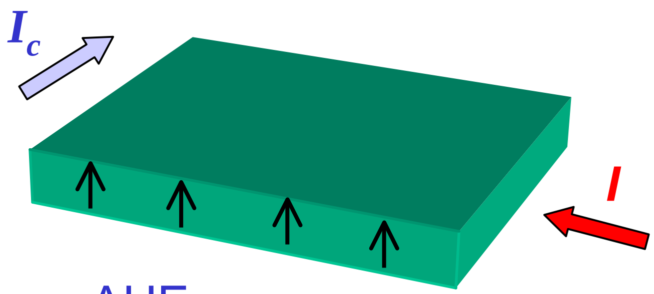


- Lorentz force is acting on electrons moving in magnetic field
- spin effects are absent
- applications (characterization of semiconductors, Hall effect sensors)

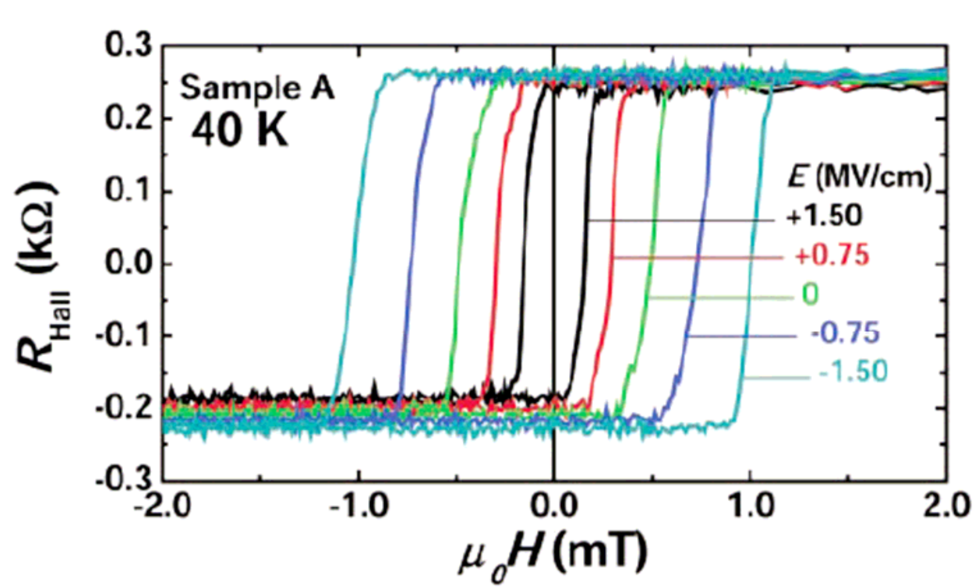


1880 – discovery of the Hall effect (E.H.Hall)
1881 – discovery of anomalous Hall effect (E.H. Hall)
1954 – first theory of the anomalous Hall effect (R. Karplus, J.M. Luttinger)
1971 – theoretical prediction of the spin Hall effect (M.I. Dyakonov, V.I. Perel)
1999 – about possible mechanism of the spin Hall effect (J.E. Hirsch, …)
2004 – observation of the spin Hall effect (Y.K. Kato et al., …)

Anomalous Hall effect



- External magnetic field is absent
- Homogeneous magnetization **M**



Measurements in magnetic field:

$$\rho_H = R_0 B + 4\pi R_s M$$

(D. Chiba et al, 2003)

Application:
- magnetic characterization
- spintronics

Mechanism of anomalous Hall effect

- Extrinsic mechanisms (related to impurities + SO interaction):
  - skew scattering (J. Smit)
  - side-jump (L. Berger)
- Intrinsic mechanism (periodic crystal potential + SO interaction)
- Chirality mechanism (in noncollinear ferromagnets)

Spin-orbit interaction

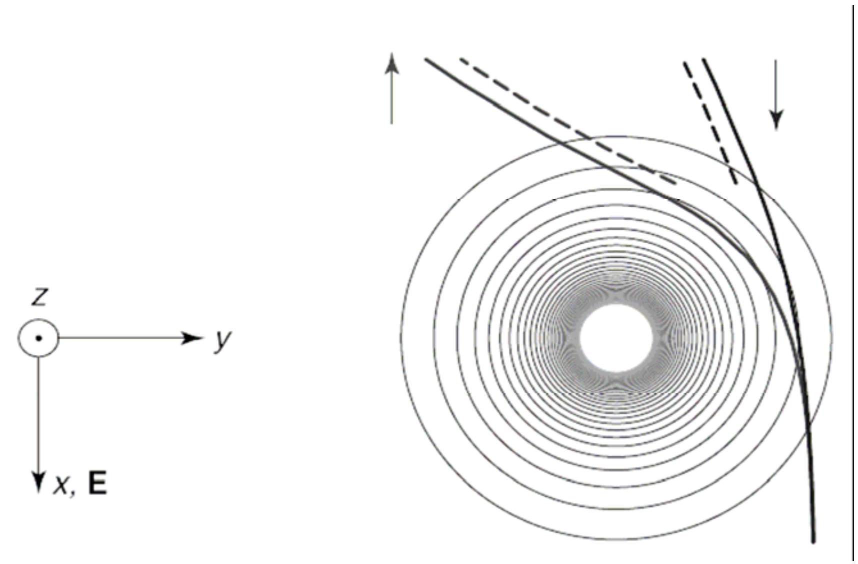
$$H_{SO,\,vac} = \lambda_{vac}\, \boldsymbol{\sigma} \cdot (\mathbf{k} \times \nabla \tilde{V})$$

In 2D high-symmetry system:

$$H_{eff} = \epsilon_k + V + H_{int} + H_{ext}$$

$$H_{int} = -\frac{1}{2}\, \mathbf{b}(\mathbf{k}) \cdot \boldsymbol{\sigma}$$

$$H_{ext} = \lambda\, \boldsymbol{\sigma} \cdot (\mathbf{k} \times \nabla V)$$

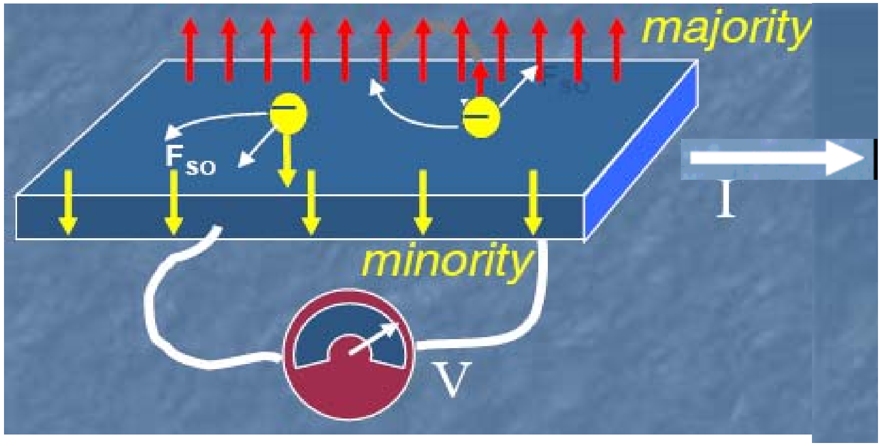


Anomalous vs. spin Hall effect

Anomalous Hall effect:

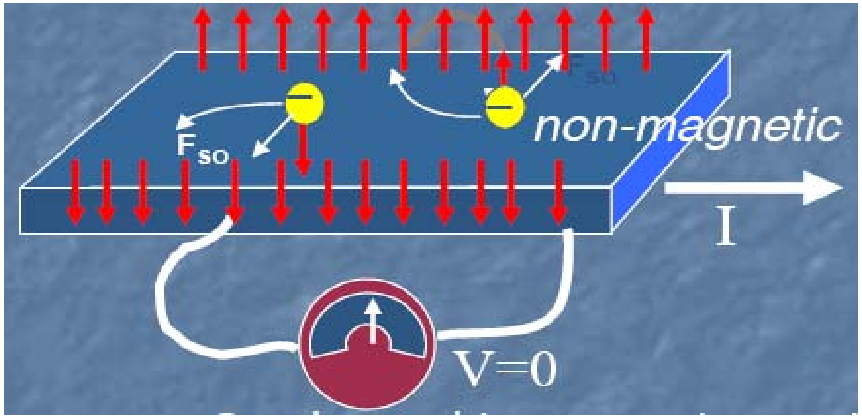
- magnetization **M**
- carriers are spin-polarized
- transverse current & spin current

  or
- Hall voltage & spin accumulation



Spin Hall effect (pure):

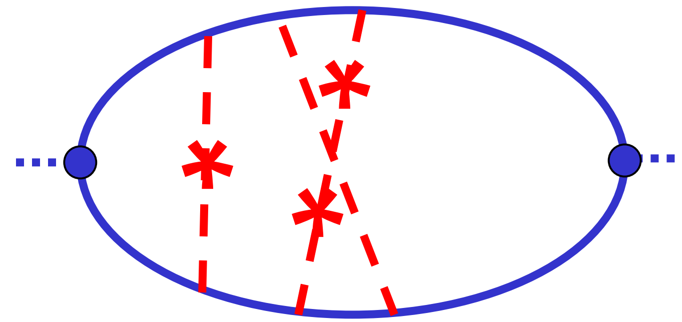
- no magnetization **M**
- carriers are not spin-polarized
- no transverse charge current
- transverse spin current
- no Hall voltage
- spin accumulation



Kubo formalism

Linear response to electric field:

$$\sigma_{ij}(\omega) = \frac{e^2}{\omega}\mathrm{Tr}\int \frac{d\varepsilon}{2\pi}\left\langle \hat{v}_i \hat{G}(\varepsilon+\omega)\hat{v}_j \hat{G}(\varepsilon)\right\rangle$$



Static limit, Středa formula:

$$\sigma_{ij} = \sigma_{ij}^{I} + \sigma_{ij}^{II}$$

$$\sigma_{ij}^{I} = \frac{e^2}{2}\mathrm{Tr}\int \frac{d\varepsilon}{2\pi}\left(-\frac{\partial f(\varepsilon)}{\partial \varepsilon}\right)\langle \hat{v}_i[\hat{G}^R(\varepsilon) - \hat{G}^A(\varepsilon)]\,\hat{v}_j\hat{G}^A(\varepsilon) - \hat{v}_i\hat{G}^R(\epsilon)\hat{v}_j[\hat{G}^R(\varepsilon) - \hat{G}^A(\varepsilon)]\rangle$$

$$\sigma_{ij}^{II} = \frac{e^2}{2}\mathrm{Tr}\int \frac{d\varepsilon}{2\pi} f(\varepsilon)\left\langle \hat{v}_i\frac{\partial \hat{G}^A(\varepsilon)}{\partial\varepsilon}\hat{v}_j\hat{G}^A(\varepsilon) - \hat{v}_i\hat{G}^A(\varepsilon)\hat{v}_j\frac{\partial \hat{G}^A(\varepsilon)}{\partial\varepsilon} + \hat{v}_i\hat{G}^R(\varepsilon)\hat{v}_j\frac{\partial \hat{G}^R(\varepsilon)}{\partial\varepsilon} - \hat{v}_i\frac{\partial \hat{G}^R(\varepsilon)}{\partial\varepsilon}\hat{v}_j\hat{G}^R(\varepsilon)\right\rangle$$

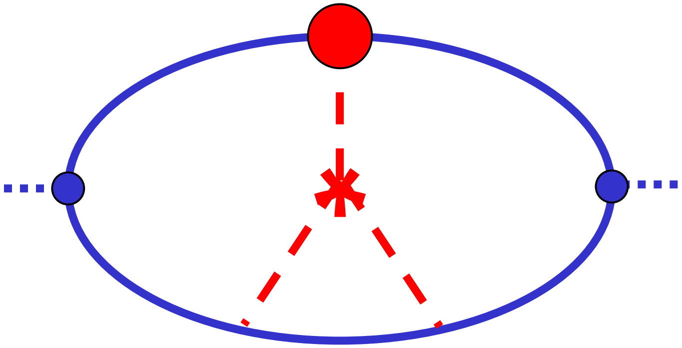
- off-diagonal conductivity contains contribution from ***all occupied states***!
- limit $\omega \to 0$ implies $\omega \ll \hbar/\tau$ *(**what is the clean limit?**)*

Skew scattering and side-jump

Skew scattering

$$\sigma_{xy}^{(SS)} = \frac{\pi e^2 \lambda_0^2}{18\hbar}\left(k_{F\downarrow}^2\, \nu_\downarrow\, v_{F\downarrow}^2\, \tau_\downarrow \frac{\nu_\downarrow \gamma_3}{\gamma_2} - k_{F\uparrow}^2\, \nu_\uparrow\, v_{F\uparrow}^2\, \tau_\uparrow \frac{\nu_\uparrow \gamma_3}{\gamma_2}\right)$$

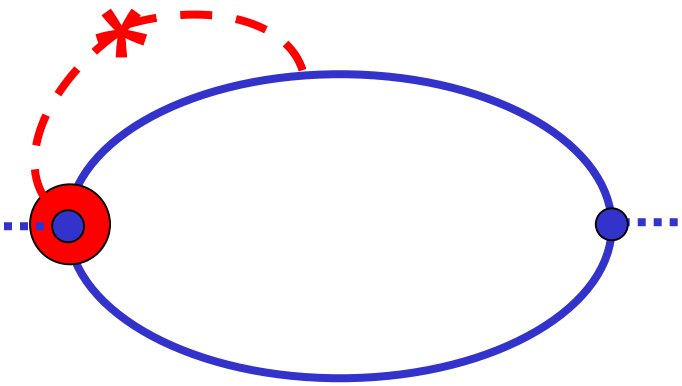
- diverges in clean limit
- vanishes in Gauss disorder potential



Side-jump

$$\sigma_{xy}^{(sj)} = \frac{e^2}{6\hbar}\lambda_0^2 (\nu_\downarrow \hbar k_{F\downarrow} v_{F\downarrow} - \nu_\uparrow \hbar k_{F\uparrow} v_{F\uparrow})$$

- does not depend on impurities



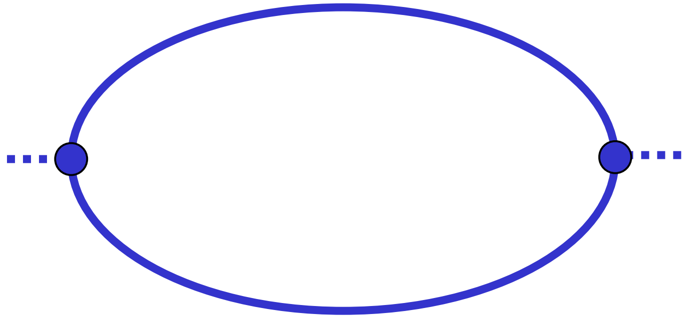
Intrinsic mechanism of AHE

- impurities are neglected
- Hamiltonian is a matrix with elements depending on **k**
- we use eigenfunctions of the Hamiltonian

Off-diagonal conductivity:

$$\sigma_{xy}(\omega) = \frac{e^2}{\omega} \int_{-\infty}^{\infty} \frac{d\varepsilon}{2\pi} \sum_{nm} \sum_{\mathbf{k}} \left(v_x\right)_{nm} G_{\mathbf{k}mm}(\varepsilon + \omega) \left(v_y\right)_{mn} G_{\mathbf{k}nn}(\varepsilon)$$

where $\mathbf{v} = \partial H / \partial \mathbf{k}$



$$\sigma_{xy} = e^2 \sum_n \sum_{\mathbf{k}} f\left(E_{\mathbf{k}n}\right) \left(\frac{\partial A_y\left(\mathbf{k}n\right)}{dk_x} - \frac{\partial A_x\left(\mathbf{k}n\right)}{dk_y} \right)$$

with gauge potential:

$$A_\alpha(\mathbf{k}n) = -i \left\langle \mathbf{k}n \left| \frac{\partial}{\partial k_\alpha} \right| \mathbf{k}n \right\rangle$$

- we obtained gauge potential in **k**-space
- AHE depends on “magnetic field” (***Berry curvature***):

$$B_\alpha(\mathbf{k}n) = \varepsilon_{\alpha\beta\gamma} \frac{\partial}{\partial k_\beta} A_\gamma(\mathbf{k}n)$$

- or on Berry phase along a contour on Fermi surface
- Fermi liquid description – with Berry curvature

Impurities are important!

2D Rashba model

Rashba Hamiltonian + magnetization:

$$H = \varepsilon_k + \alpha\left(k_y\sigma_x - k_x\sigma_y\right) - M\sigma_z, \quad \varepsilon_k = k^2/2m$$

Hall conductivity

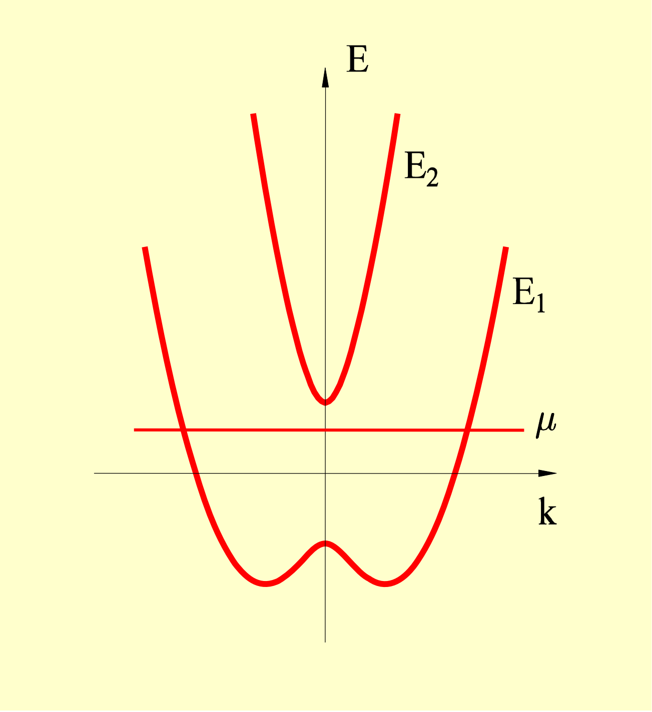
$$\sigma_{xy}(\omega) = \frac{e^2}{\omega}\mathrm{Tr}\int_{-\infty}^{\infty}\frac{d\varepsilon}{2\pi}\frac{d^2\mathbf{k}}{(2\pi)^2} v_x\, G_{\mathbf{k}}(\varepsilon+\omega)\, v_y\, G_{\mathbf{k}}(\varepsilon)$$

where

$$v_x = \frac{k_x}{m} - \alpha\,\sigma_y \qquad v_y = \frac{k_x}{m} + \alpha\,\sigma_x$$

Energy spectrum:

$$E_{\mathbf{k}\uparrow,\downarrow} = \varepsilon_k \mp \lambda(k), \quad \lambda(k) = \sqrt{M^2 + \alpha^2 k^2}$$



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Two contributions:

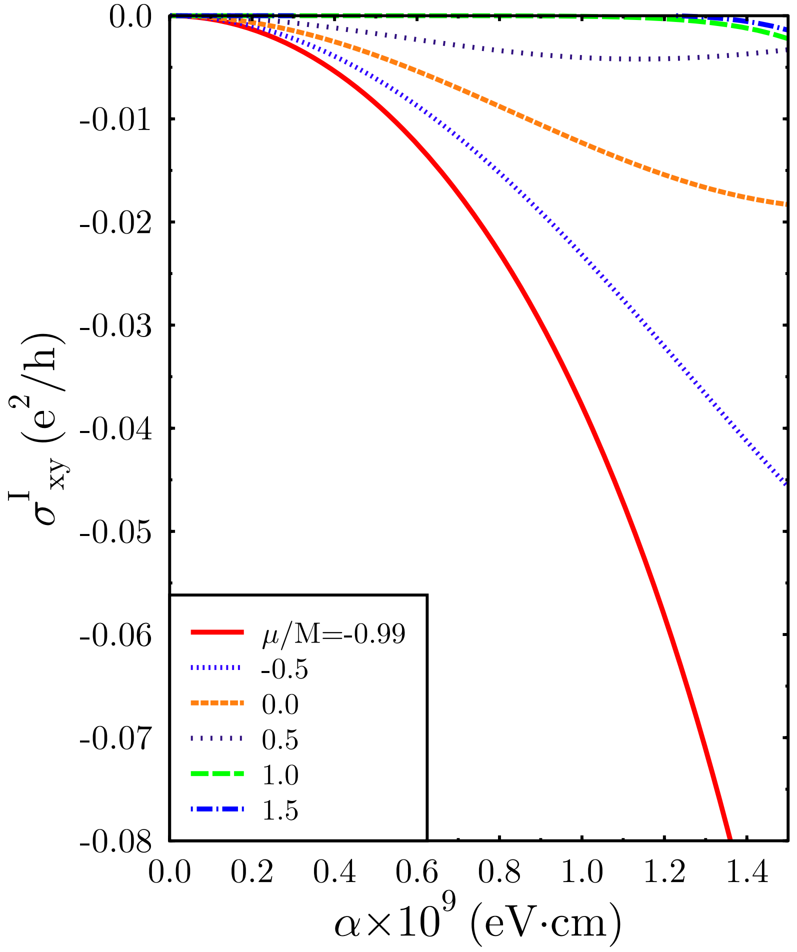
$$\sigma_{xy} = \sigma_{xy}^{I} + \sigma_{xy}^{II}$$

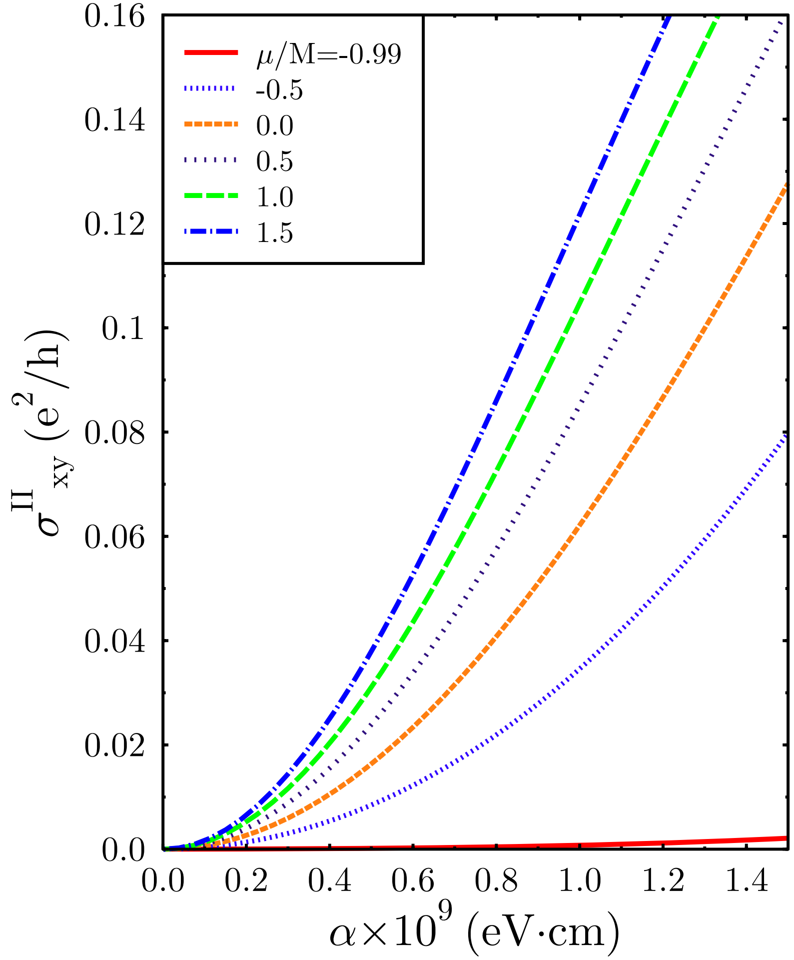
Contribution of states below Fermi energy:

$$\sigma_{xy}^{II} = -4e^2 M \alpha^2 \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{f\left(E_{\mathbf{k}\uparrow}\right) - f\left(E_{\mathbf{k}\downarrow}\right)}{\left(E_{\mathbf{k}\uparrow} - E_{\mathbf{k}\downarrow}\right)^3}$$

$$\Rightarrow \quad \sigma_{xy}^{II} = \frac{e^2 M}{4\pi} \left(\frac{1}{\lambda\left(k_{F\downarrow}\right)} - \frac{1}{\lambda\left(k_{F\uparrow}\right)} \right)$$

Different contributions to Hall conductivity





Topology

Hamiltonian:

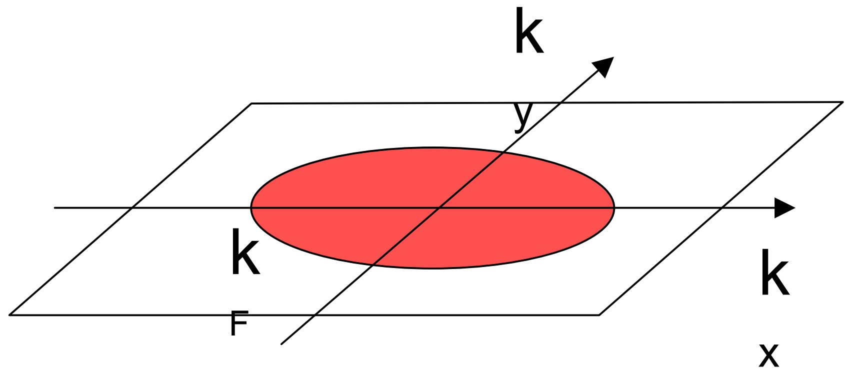
$$H = \varepsilon_{\mathbf{k}} + \lambda(k)\,\boldsymbol{\sigma}\cdot\mathbf{n}(\mathbf{k})$$

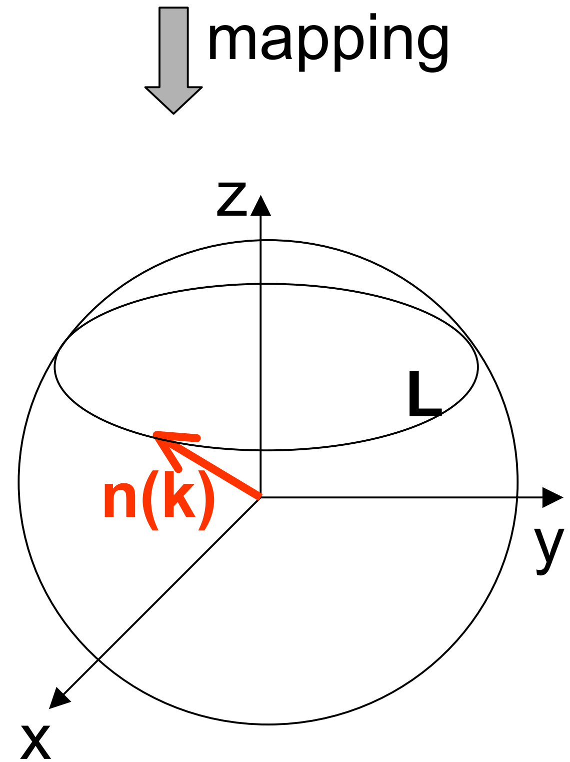
where the unit vector

$$\mathbf{n}(\mathbf{k}) = \left(\frac{\alpha k_y}{\lambda(k)},\quad -\frac{\alpha k_x}{\lambda(k)},\quad -\frac{M}{\lambda(k)} \right)$$

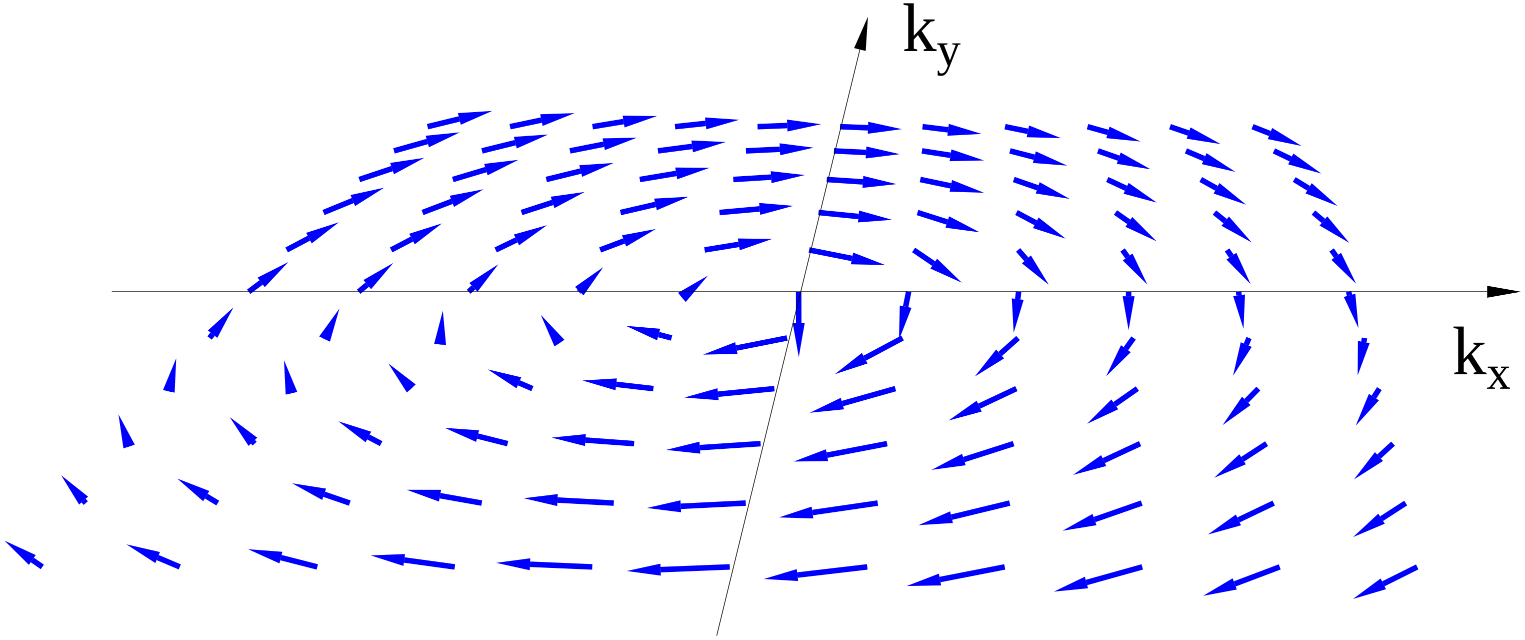
In terms of **n**(**k**):

$$\sigma_{xy}^{II} = -\frac{e^2}{2}\int \frac{d^2\mathbf{k}}{(2\pi)^2}\, f\left(E_{\mathbf{k}\uparrow}\right)\varepsilon_{\alpha\beta\gamma}\, n_\alpha \frac{\partial n_\beta}{\partial k_x}\frac{\partial n_\gamma}{\partial k_y}$$



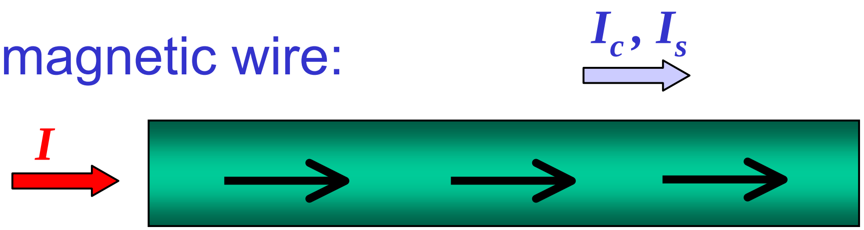


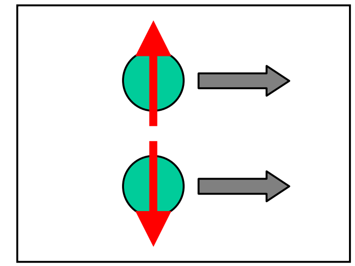
n-field in 2D case



Spin current

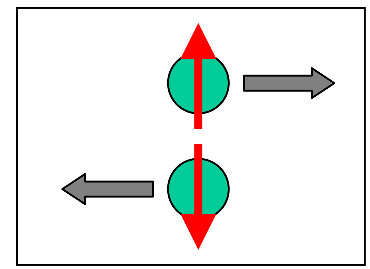
Current in magnetic wire:





Spin current can be defined as

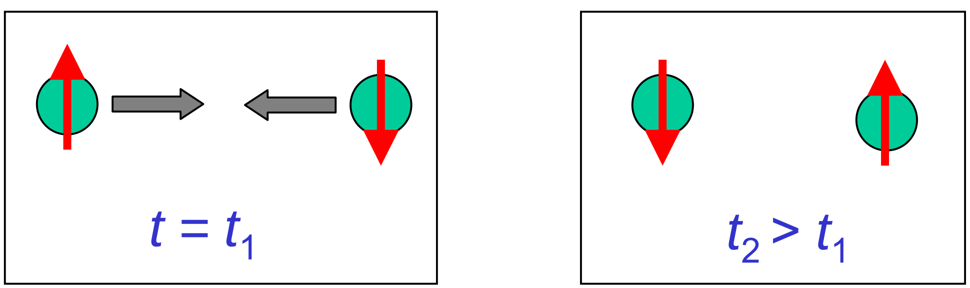
$$j_k^i \equiv \frac{1}{2} n \langle \sigma_i v_k + v_k \sigma_i \rangle$$

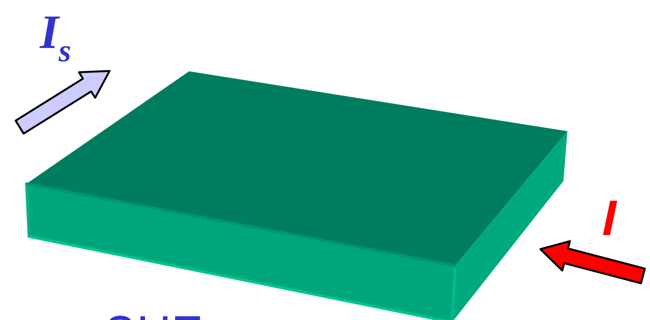


pure spin current

- it allows equilibrium spin current
- spin current is not conserved

Is the spin current related to real motion?





SHE

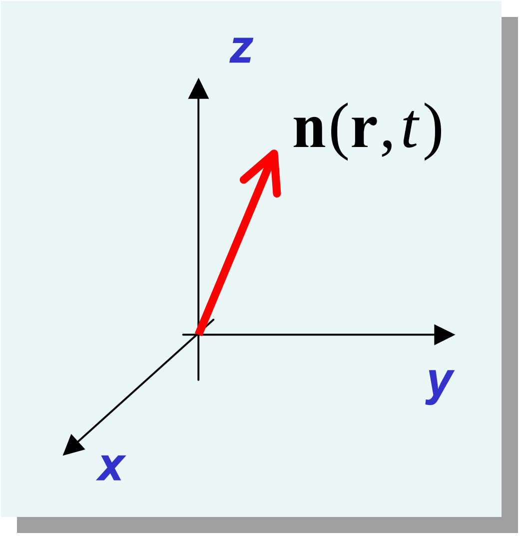
Spin current is ill defined?

Definition of spin current

Lagrangian:

$$L = \int d^3\mathbf{r}\, \psi^+(\mathbf{r},t) \left(i\frac{\partial}{\partial t} - H \right) \psi(\mathbf{r},t)$$

$$L = \int d^3\mathbf{r}\, \psi^\dagger(\mathbf{r},t) \left(i\,\frac{\partial}{\partial t} + \frac{\nabla^2}{2m} - V(\mathbf{r}) \right) \psi(\mathbf{r},t),$$



Local transformations (rotations) in the spin space:

$$\psi(\mathbf{r},t) \longrightarrow \exp\left[-ig\,\mathbf{n}(\mathbf{r},t)\cdot\boldsymbol{\sigma}\right]\psi(\mathbf{r},t),$$

Transformed Lagrangian:

$$L = \int d^3\mathbf{r}\, \psi^\dagger(\mathbf{r},t) \left[i\left(\frac{\partial}{\partial t} - iA_0^i(\mathbf{r},t)\,\sigma^i \right) + \frac{1}{2m}\left(\frac{\partial}{\partial r_\alpha} - iA_\alpha^i(\mathbf{r},t)\,\sigma^i \right)^2 - V(\mathbf{r}) \right] \psi(\mathbf{r},t),$$

Gauge fields:

$$A_0^i(\mathbf{r},t) = g\,\frac{\partial n^i(\mathbf{r},t)}{\partial t}, \qquad A_\alpha^i(\mathbf{r},t) = g\,\frac{\partial n^i(\mathbf{r},t)}{\partial r_\alpha}.$$

Spin density and spin current density:

$$S^i(\mathbf{r},t) = \frac{\delta L}{\delta A_0^i(\mathbf{r},t)}\,, \qquad J_\alpha^i(\mathbf{r},t) = \frac{\delta L}{\delta A_\alpha^i(\mathbf{r},t)}\,.$$

Equilibrium spin current

Toy model: two spins in local fields

Hamiltonian with local fields:

$$H = -J\mathbf{S}_1 \cdot \mathbf{S}_2 - \mathbf{B}_1 \cdot \mathbf{S}_1 - \mathbf{B}_2 \cdot \mathbf{S}_2 .$$

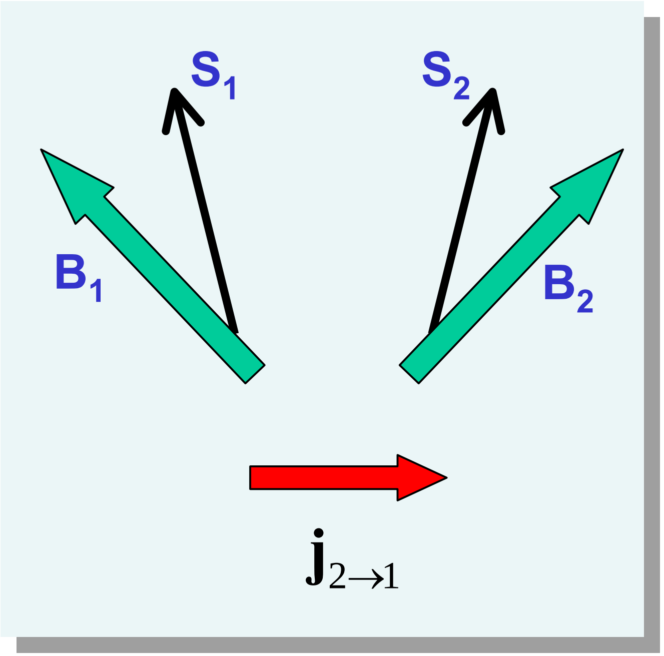
Equation of motion for spin S_1:

$$\dot{S}_{1\mu} = (i/\hbar)[H, S_{1\mu}]$$

$$\dot{\mathbf{S}}_1 = (J/\hbar)\mathbf{S}_1 \times \mathbf{S}_2 + (1/\hbar)\mathbf{S}_1 \times \mathbf{B}_1$$

Spin current

$$\mathbf{j}_{2\to 1} \equiv \frac{J}{\hbar}\mathbf{S}_1 \times \mathbf{S}_2 = -\mathbf{j}_{1\to 2},$$



Hubbard model (on-site e-e interactions)

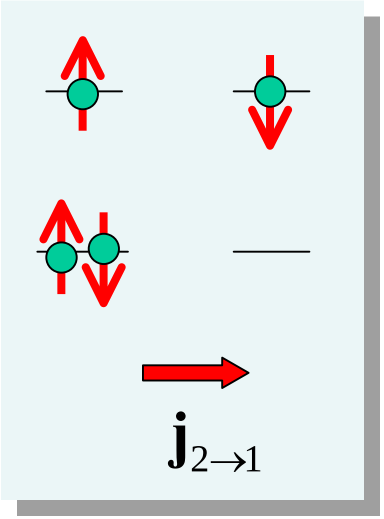
Hubbard Hamiltonian for two sites:

$$H = -t(c^\dagger_{1\alpha}c_{2\alpha} + c^\dagger_{2\alpha}c_{1\alpha}) + \sum_{i=1,2}(Un_i^\uparrow n_i^\downarrow - \mathbf{B}_i\cdot\mathbf{S}_i),$$

Equation of motion:

$$\dot{\mathbf{S}}_1 = (i/\hbar)[H,\mathbf{S}_1].$$

$$\dot{\mathbf{S}}_1 = \frac{it}{2\hbar}(c^\dagger_{1\alpha}c_{2\beta} - c^\dagger_{2\alpha}c_{1\beta})\boldsymbol{\sigma}_{\alpha\beta} + \frac{1}{\hbar}\mathbf{S}_1\times\mathbf{B}_1$$



Spin current:

$$\mathbf{j}_{2\to 1} \equiv \frac{it}{2\hbar}(c^\dagger_{1\alpha}c_{2\beta} - c^\dagger_{2\alpha}c_{1\beta})\boldsymbol{\sigma}_{\alpha\beta} = -\mathbf{j}_{1\to 2}$$

Equilibrium spin current is related with hopping between different sites

Basis functions: $|\uparrow\downarrow,0\rangle,\ |0,\uparrow\downarrow\rangle,\ |\uparrow,\uparrow\rangle,\ |\uparrow,\downarrow\rangle,\ |\downarrow,\uparrow\rangle,\ |\downarrow,\downarrow\rangle$

Strong e-e interaction: $t/U \ll 1$

Effective Hamiltonian:

$$\widetilde{H} = -J(\mathbf{S}_1 \cdot \mathbf{S}_2 - 1/4) - \mathbf{B}_1 \cdot \mathbf{S}_1 - \mathbf{B}_2 \cdot \mathbf{S}_2 \qquad -J \equiv 4t^2/U$$

Spin current:

$$\widetilde{\mathbf{j}}_{2\to 1} \equiv \frac{J}{\hbar}\mathbf{S}_1 \times \mathbf{S}_2,$$

Spin Hall effect from anomalous Hall effect

Anomalous Hall voltage produced by spin up electrons

$$V_H = 4R_s L j_x n_\uparrow \mu_B$$

Spin Hall voltage

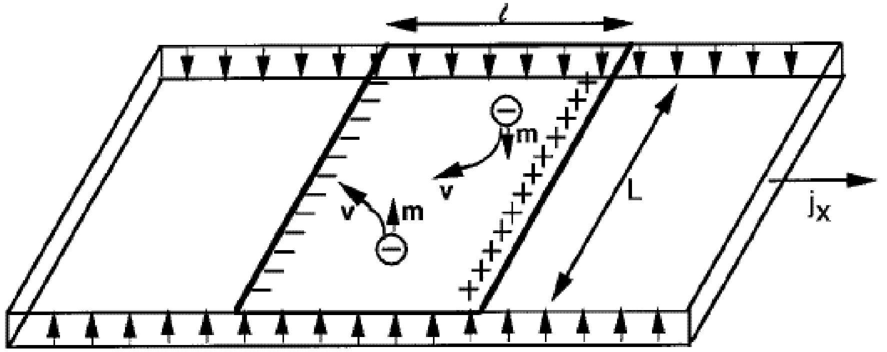
$$V_{\mathrm{SH}} = 2\pi R_s L j_x n \mu_B$$

Spin current for each spin

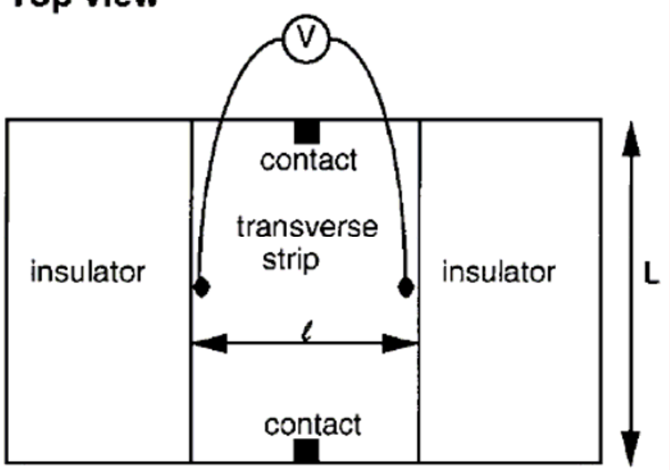
$$j_\sigma = V_{\mathrm{SH}} \rho L$$

Voltage due to spin current

$$V_{\mathrm{SC}} = 8\pi^2 R_s^2 l \frac{(n\mu_B)^2}{\rho} j_x .$$

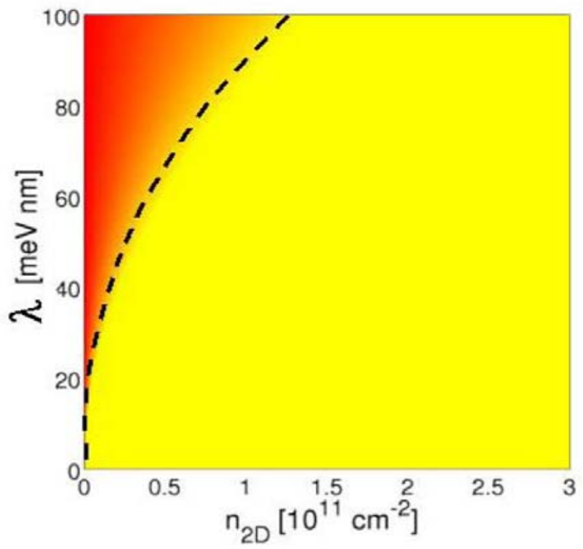


Top view



(Hirsch, 1999)

Universal spin-Hall conductivity



Model assumptions:

- 2D Rashba model
- no impurities

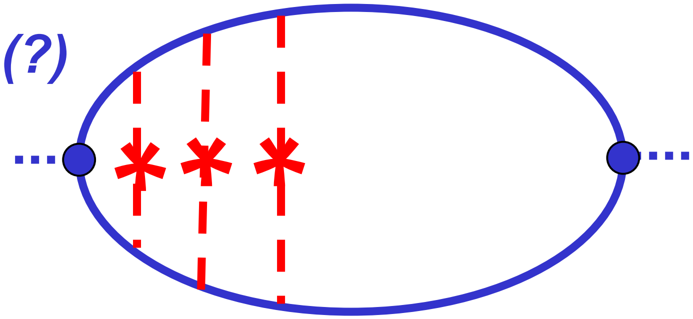
Intrinsic contribution to spin Hall effect:

$$\sigma_{xy}^{SH} = \frac{e}{8\pi}$$ (J. Sinova et al, 2004)

When impurities are taken into account, and $N_{imp} \rightarrow 0$

$$\sigma_{xy}^{SH} = 0$$

vertex correction:



This cancellation is special for Rashba model (?)

Quantized AHE and SHE

2D Dirac model (graphene)

$$H = v\left(k_x\sigma_x + k_y\sigma_y\right) + \Delta\sigma_z$$

Energy spectrum

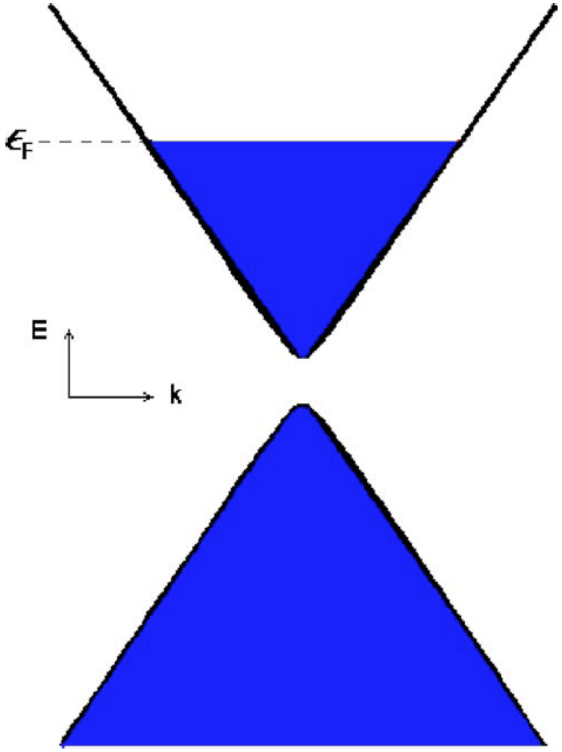
$$\varepsilon = \pm E_k, \quad E_k = \sqrt{\Delta^2 + v^2k^2}$$

Intrinsic Hall conductivity

$$\sigma_{xy} = -\frac{e^2}{4\pi}\frac{\Delta}{E_F}, \quad E_F > \Delta$$

$$\sigma_{xy} = -\frac{e^2}{4\pi}, \quad E_F \text{ in the gap}$$

$$\sigma_{xy}^{SH} = \frac{2}{e}\sigma_{xy}$$



Conclusions

- There are extrinsic (impurity) and intrinsic (topology) mechanisms of AHE and SHE both due to the SO interaction
- The intrinsic mechanism is related to the topology of the energy bands
- Intrinsic mechanism is also strongly affected by impurities
- Effect of impurities is ***stronger in the clean limit*** leading to possible compensation of the intrinsic mechanism of AHE and SHE
- Equilibrium spin currents can be additionally generated in inhomogeneous magnetic systems

In the 2D case:

$$\sigma_{xy}^{II} = \frac{e^2}{2} \sum_n \sum_{\mathbf{k}} f\left(E_{\mathbf{k}n}\right) \varepsilon_{ij} F_{ij}$$

where “gauge field tensor”:

$$F_{ij} = \frac{\partial A_j}{\partial k_i} - \frac{\partial A_i}{\partial k_j}$$

Eigenvector:

$$\left|\mathbf{k}\uparrow\right\rangle = \sqrt{\frac{M+\lambda(k)}{2\lambda(k)}} \begin{pmatrix} 1 \\ \dfrac{i\alpha\left(k_x + ik_y\right)}{M+\lambda(k)} \end{pmatrix}$$

$$\mathbf{A}(\mathbf{k}) = \left(-\frac{\alpha^2 k_y}{2\lambda(k)\left[M+\lambda(k)\right]},\quad \frac{\alpha^2 k_x}{2\lambda(k)\left[M+\lambda(k)\right]} \right)$$